A CERTAIN CLASS OF JENSEN MEASURES FOR UNIFORM ALGEBRAS

BY
CHO-ICHIRO MATSUOKA
Faculty of Engineering, Doshisha University, Kyoto, 602 Japan

ABSTRACT
For any uniform algebra A and any point q of the maximal ideal space of A there exists a Jensen measure λ for q carried on the Shilov boundary for A such that λ admits the generalized Brownian maximal function to each nonnegative A-subharmonic function in $C_R(X)$. The maximal function and its original function satisfy Doob's inequality, Burkholder–Gundy–Silverstein inequalities and Fefferman–Stein inequality.

1. Introduction

Let X be the maximal ideal space of an arbitrary uniform algebra A and let q be any point of X. The letter J denotes the totality of continuous A-subharmonic funtion on X (cf. [6]). The convex cone J defines the partial order on the totality $M^+(X)$ of finite positive regular Borel measures on X. This order relation is denoted by the symbol $\prec$, i.e. the relation $\tau \prec \nu$ holds for any pair τ, ν in $M^+(X)$ if and only if they satisfy $\int k d\tau \leqq \int k d\nu$ for each k of J. Also we note that a positive measure τ supported on X is a Jensen measure for $q \in X$ if and only if it satisfies the relation $\delta_q \prec \tau$, where δ_q is the Dirac measure at q.

In [8], we studied fundamental inequalities among p-th means of functions in J with respect to a certain type of Jensen measure λ_x for q. The measure λ_x was assumed to be the distinguished member in the following subclass of $M^+(X)$. Let $\mathscr{F}_q$ be the totality of the compact subsets of X that contain q. We use the letter $\tilde{\mathscr{D}}$ to denote a subset of $\mathscr{F}_q$ which includes X.

DEFINITION 1.1 (cf. [8]). A subfamily $\{\lambda_G : G \in \tilde{\mathscr{D}}\}$ of $M^+(X)$ indexed by $\tilde{\mathscr{D}}$ is called a consistent family with base point q, if it satisfies

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(1) each λ_G is carried on G and satisfies $\delta_q \prec \lambda_G$;

(2) if $F \subset G$, $F, G \in \tilde{\mathscr{D}}$, then $\lambda_F \prec \lambda_G$;

(3) if $F \subset G$ and if U is the relative interior of F with respect to G, then the restriction $\lambda_F \mid U$ of λ_F onto U is absolutely continuous with respect to λ_G and its density satisfies

$$0 \leqq d(\lambda_F \mid U)/d\lambda_G \leqq 1 \quad \text{a.e. } \lambda_G;$$

(*) the maximum measure λ_X in the family with respect to the order $\prec$ is called the terminal measure of the family.

Throughout this paper we are mainly interested in consistent families which are indexed by $\mathscr{F}_q$. The notion of consistent family was required to generalize the probabilistic theory of Hardy spaces in the context of a uniform algebra. We used such a family as a substitute for Brownian motion, which has been one of the powerful devices in the analysis of classical Hardy spaces. In concrete algebras such as the disk algebra $A(D)$, two dimensional Brownian motion starting at 0 combines with the consistent family via the stopping time argument: A. Debiard and B. Gaveau [3] established that the first exit time of the Brownian paths with respect to a compact subset $G \ni 0$ of $\bar{D}$ yields the Keldysh measure for 0 supported on the boundary of G. The totality of these Keldysh measures was the original model of consistent families, and in this case the terminal measure is identical with the normalized Lebesgue measure on the circle. The most interesting observation in this line is that the Brownian maximal function of nonnegative function on $\bar{D}$ can be obtained directly from Keldysh measures, as long as we are interested in its conditional expectation relative to Baire σ-algebra on the circle. We can complete the above translation without applying probabilistic argument. Furthermore, in that procedure, only three properties of Keldysh measures are needed, that is, the conditions in Definition 1.1. Thus we reached the notion of consistent family in the context of a uniform algebra. It was proven in [8] that the terminal measure of any consistent family indexed by $\mathscr{F}_q$ admits the maximal function, just like the conditional expectation of Brownian maximal function, to each nonnegative function in J. The maximal function and its original function satisfy Doob's maximal inequality [4], Burkholder–Gundy–Silverstein inequalities [2], Fefferman–Stein inequality [5]. Also it is not so hard to prove Zygmund's $L \log L$ estimate (cf. [10]) etc., though we did not discuss them in [8]. Hence so far as basic results on Hardy spaces are concerned, the notion of consistent family may be regarded as a useful substitute for Brownian motion. But, if we want to justify this viewpoint more explicity, we must answer the important question: to what extent do consistent families exist? The purpose here is to present some information about this problem. Our main result is as follows. For an arbitrary uniform algebra A and any point q of the maximal ideal space X of A there exists a consistent family of Jensen measures, indexed by $\mathscr{F}_q$, whose terminal measure is carried on the Shilov boundary for A (Theorem 3.1). The key to our study is the localization principle due to T. W. Gamelin and N. Sibony [7], which guarantees the sheaf structure of the cone of A-subharmonic functions on X.

Here some comment should be made about the style of this article. Although our concern is focussed on uniform algebras, the principal results in this paper will be described in a more general situation. It is mainly because we want to treat Haar measures on tori in connection with the cone of Rudin's n-subharmonic functions on a polydisk. Furthermore, to present various examples of general consistent families, we shall investigate the probability measures on compact convex sets (Section 4). It will be proven that such a measure is always the terminal measure of a general consistent family.

Finally the author should like to express his deep gratitude to Professor T. W. Gamelin for his encouragement during this study.

2. Existence theorem for consistent families

Throughout this section we shall assume that X is an arbitrary, but fixed, compact Hausdorff space. The letter J denotes the *convex subcone* of $C_R(X)$ such that

(a) J contains the constants R and separates the points on X;

(b) J is stable in the max-operation, i.e. the function $(f \vee g)(x) = \max\{f(x), g(x)\}$ is contained in J provided f, g are in J;

(c) J has the sheaf structure. In other words, for any open cover $\{U_i\}_{i \in I}$ of X and any set $\{f_i\}_{i \in I}$ of functions in J, if a function $g \in C_R(X)$ satisfies $g = f_i$ on each U_i, then g belongs to J.

We first clarify the notations and mention the fundamental results about J, most of which can be found in A. M. Alfsen [1] and T. W. Gamelin [6]. Let f be an extended real-valued function defined on a subset E of X. The lower J-envelope $\check{f}$ of f is the function on X defined as

$$\check{f}(x) = \sup\{g(x) : g \in J, g \,|\, E \leqq f\}, \qquad \forall x \in X.$$

The partial order on $M^+(X)$ induced by J is denoted by the symbol $\prec$. A positive measure is said to be maximal on a compact subset G of X, if it is maximal in $M^+(G)$ with respect to the order $\prec$. It is know that any positive measure on G is dominated, with respect to the order $\prec$, by a measure maximal on G. Furthermore a positive measure μ carried on G is maximal on G, if and only if it satisfies $f = \check{f}$ a.e. μ, $\forall f \in C_R(G)$. Let p be any point of X for which the Dirac measure δ_p is maximal on X. The totality of such points forms the Choquet boundary for J, whose closure is known as the J-Shilov boundary. The J-Shilov boundary is the minimum closed subset of X on which every function in J attains its maximum value on X. Applying the Bishop–de Leeuw theorem, we find that any positive measure maximal on X is carried on the J-Shilov boundary. Since J is assumed to have the sheaf structure, the positive measures maximal on X have the following localization property.

THEOREM 2.1. *Suppose a positive measure μ is carried on the interior* Int G *of a compact set G in X. If μ is maximal on G, then μ is maximal on X.*

PROOF. It suffices to show that $\int f d\mu = \int \check{f} d\mu$ for every $f \in C_R(X)$. We may suppose f is nonnegative, because J contains the constants R. Then for any strictly positive number ε, there exists a nonnegative function g of $C_R(X)$, compactly supported on Int G, such that $g \leqq f$ and $\int g d\mu \geqq \int f d\mu - \varepsilon$. This is due to the fact that μ is carried on Int G. On the other hand, if $h \in J$ is dominated by g on G, so is the function h^* which is equal to $h \vee 0$ on G and equal to 0 outside G. By the sheaf structure of J, h^* belongs to J. This implies that $\check{g} = g \check{\restriction} G$ on G. Since μ is maximal on G, we have

$$\int f d\mu \geqq \int \check{f} d\mu \geqq \int \check{g} d\mu = \int g d\mu > -\varepsilon + \int f d\mu.$$

Letting ε go to 0, we conclude that $\int f d\mu = \int \check{f} d\mu$.

Let q be an arbitrary point of X and let $\mathscr{F}_q$ be the totality of compact subsets of X, each of which contains q. The letter $\mathscr{D}$ denotes a subset of $\mathscr{F}_q$. Let $\{\lambda_G : G \in \mathscr{D}\}$ be a collection of positive measures supported on X. This family in $M^+(X)$ is called a consistent family with base point q, if it satisfies three conditions in Definition 1.1. If further $\mathscr{D}$ contains X, the element λ_X of the family will be refered to as the terminal measure. The following stability properties of consistent families are easy to check (cf. Theorem 4.8, 4.9 in [8]).

THEOREM 2.2. *For two consistent families $(\{\lambda_G^i\}_{G \in \mathscr{D}})_{i=1,2}$ with base point q, their convex combination $\{s\lambda_G^1 + (1-s)\lambda_G^2 : G \in \mathscr{D}\}$, $0 \leqq s \leqq 1$, is a consistent family with base point q.*

Let $(\{\lambda_G^i\}_{G\in\mathscr{D}})_{i\in I}$ be a set of consistent families, indexed by a directed set I, whose base points are identical. Suppose, for each $G \in \mathscr{D}$, $(\lambda_G^i)_{i\in I}$ converges vaguely to λ_G along I. Then $\{\lambda_G : G \in \mathscr{D}\}$ is a consistent family with common base point.

From now on we set about to construct a consistent family, indexed by $\mathscr{F}_q$, whose terminal measure is carried on the J-Shilov boundary. We first manufacture a finite model of diffusion processes such as the Brownian motion on the plane. For this aim we must define the channels along which our finite model flows.

Definition 2.3. Let $\mathscr{D}$ be any *finite* subset of $\mathscr{F}_q$. A path bundle Q on $\mathscr{D}$ is a finite sequence $Q = \{(\mathscr{B}_j, \mathscr{C}_j)\}_{j=1}^n$ of the pairs $(\mathscr{B}_j, \mathscr{C}_j)$, $1 \leqq j \leqq n$ such that

(1) $\mathscr{B}_1 = \mathscr{D}$;

(2) $\mathscr{C}_j \subset \mathscr{B}_j \subset \mathscr{D}$, $1 \leqq j \leqq n$ (possibly $\mathscr{C}_j = \mathscr{B}_j$ or $\mathscr{B}_j = \mathscr{D}$);

(3) $\mathscr{B}_{k+1} = \mathscr{B}_k \setminus \mathscr{C}_k$ and $\mathscr{C}_k \neq \varnothing$ if $k < n$.

The totality of the path bundles on $\mathscr{D}$ is denoted by $\Omega(\mathscr{D})$. We list the notations needed in the subsequent argument.

(4) $\operatorname{card}(Q)$ = the cardinal number of a set Q;

(5) $Q \mid k = \{(\mathscr{B}_j, \mathscr{C}_j)\}_{j\leqq k}$, where $Q = \{(\mathscr{B}_j, \mathscr{C}_j)\}_{j=1}^n$ and $k \geqq 1$;

(6) $(\mathscr{B}(Q), \mathscr{C}(Q))$ = the final term of Q;

(7) $E(Q) = [\bigcap(\operatorname{Int} V : V \in \mathscr{B}(Q)\setminus\mathscr{C}(Q)\}] \cap [\bigcap\{\partial V : V \in \mathscr{C}(Q)\}]$ (we put, for convenience $X = \bigcap\{\operatorname{Int} V : V \in \varnothing\} = \bigcap\{\partial V : V \in \varnothing\}$).

(8) $Q + 1 = \{Q^* \in \Omega(\mathscr{D}) : Q \subset Q^*, \mathscr{B}(Q^*) = \mathscr{B}(Q)\setminus\mathscr{C}(Q)\}$.

The following properties of $\Omega(\mathscr{D})$ are easy to check.

Proposition 2.4. *For any finite subset $\mathscr{D}$ of $\mathscr{F}_q$, the path bundles Q, Q^* on $\mathscr{D}$ satisfy*

(1) *the inclusion $Q \subset Q^*$ implies $Q = Q^* \mid \operatorname{card}(Q)$;*

(2) *$Q \mid k \in \Omega(\mathscr{D})$ whenever $Q \in \Omega(\mathscr{D})$ and $k \geqq 1$;*

(3) *in the case $Q^* \in Q + 1$, $\operatorname{card}(Q^*) = 1 + \operatorname{card}(Q)$ if $\mathscr{C}(Q)$ is nonempty, and otherwise $Q + 1$ is a singleton $\{Q\}$;*

(4) *if $\mathscr{C}(Q) \neq \varnothing$, then $\{E(Q^*) : Q^* \in Q + 1\}$ gives a partition of the set $\bigcap\{V : V \in \mathscr{B}(Q)\setminus\mathscr{C}(Q)\}$ into pairwise disjoint Borel sets in X (note that $\mathscr{B}(Q^*) = \mathscr{B}(Q)\setminus\mathscr{C}(Q)$).*

Our finite model of diffusion processes is defined as a certain type of map from $\Omega(\mathscr{D})$ into $M^+(X)$, which will be called a tree along $\Omega(\mathscr{D})$.

Definition 2.5. Let $\mathscr{D}$ be a finite subset of $\mathscr{F}_q$. A map $\lambda : \Omega(\mathscr{D}) \to M^+(X)$

from $\Omega(\mathscr{D})$ into $M^+(X)$ is called a tree along $\Omega(\mathscr{D})$ with base point $q \in X$ if it satisfies

(1) each $\lambda(Q)$ is carried on the closure $\overline{E(Q)}$ in X of $E(Q)$, especially $\lambda(Q) = 0$ if $E(Q) = \varnothing$;

(2)
$$\delta_q \prec \sum_{\substack{Q \in \Omega(\mathscr{D}) \\ \mathrm{card}(Q) = 1}} \lambda(Q);$$

(3) for any $Q \in \Omega(\mathscr{D})$, $\lambda(Q) \prec \Sigma_{Q^* \in Q+1} \lambda(Q^*)$.

PROPOSITION 2.6. *For each finite subset $\mathscr{D} \subset \mathscr{F}_q$ there exists a tree λ along $\Omega(\mathscr{D})$, with base point q, which satisfies the stronger version* (1*) *of* (1) *in Definition* 2.5:

(1*) *each $\lambda(Q)$ is carried on $E(Q)$ and is maximal on the closed set $\bigcap\{V : V \in \mathscr{B}(Q)\}$ in the order $\prec$ (here we put $X = \bigcap\{V : V \in \varnothing\}$).*

PROOF. Call $\Omega \mid n = \{Q \in \Omega(\mathscr{D}) : \mathrm{card}(Q) \leqq n\}$ and by induction we construct a map $\lambda_n : \Omega \mid n \to M^+(X)$ which has the properties (1*), (2) of Definition 2.5 and

(3*) if Q of $\Omega \mid n$ admits the inclusion $Q + 1 \subset \Omega \mid n$, then

$$\lambda_n(Q) \prec \sum_{Q^* \in Q+1} \lambda_n(Q^*).$$

Since the length of the path bundles in $\Omega(\mathscr{D})$ is uniformly bounded, any λ_n with n large enough will work as the desired map λ.

We first take a measure μ from $M^+(X)$ which is maximal on the closed set $\bigcap\{V : V \in \mathscr{D}\}$ and satisfies $\delta_q \prec \mu$. Let $\lambda_1 : \Omega \mid 1 \to M^+(X)$ be the map defined by $\lambda_1(Q) = \mu \mid E(Q)$, $\forall Q \in \Omega \mid 1$. Since the collection $\{E(Q);\ Q \in \Omega(\mathscr{D}),\ \mathrm{card}(Q) = 1\}$ gives a partition of the set $\bigcap\{V : V \in \mathscr{D}\}$ into pairwise disjoint Borel sets, λ_1 is surely one of the desired maps.

Next suppose that a map $\lambda_n : \Omega \mid n \to M^+(X)$ has (1*) (2), (3*). For each $Q \in (\Omega \mid n) \backslash (\Omega \mid n-1)$ there exists $\mu_Q \in M^+(X)$ which is maximal on the set $\bigcap\{V : V \in \mathscr{B}(Q) \backslash \mathscr{C}(Q)\}$ and such that $\lambda_n(Q) \prec \mu_Q$. Note that if $\mathscr{C}(Q) = \varnothing$ or equivalently if $Q + 1 = \{Q\}$ ($\subset \Omega \mid n$), we have $\lambda_n(Q) = \mu_Q$; it is because $\lambda_n(Q)$ is maximal on $\bigcap\{V : V \in \mathscr{B}(Q)\}$ by the induction hypothesis. Here we require that the collection $\{Q + 1 : Q \in (\Omega \mid n) \backslash (\Omega \mid n-1),\ \mathscr{C}(Q) \neq \varnothing\}$ gives a partition of $(\Omega \mid n+1) \backslash (\Omega \mid n)$ into pairwise disjoint subsets. Indeed, for any $Q \in (\Omega \mid n+1) \backslash (\Omega \mid n)$, relations $Q \mid n \in \Omega \mid n$ and $\mathscr{C}(Q \mid n) \neq \varnothing$ hold by the very definition of the path bundles. In particular, $Q \in (Q \mid n) + 1$. On the other hand if $(Q_1+1)\cap(Q_2+1)\ni Q$ for some Q_1, Q_2 of $(\Omega\,|\,n)\setminus(\Omega\,|\,n-1)$, then the equality $Q_1 = Q\,|\,n = Q_2$ must be valid. These yield the above fact. Furthermore, we get

$$\mu_Q = \sum_{Q^*\in Q+1} \mu_Q\,|\,E(Q^*) \qquad \forall Q\in(\Omega\,|\,n)\setminus(\Omega\,|\,n-1).$$

This is due to the fact that $\{E(Q^*): Q^*\in Q+1\}$ gives a decomposition of the set $\bigcap\{V: V\in\mathscr{B}(Q)\setminus\mathscr{C}(Q)\}$ into pairwise disjoint Borel sets whenever $\mathscr{C}(Q)$ is nonempty. Put for each $Q^*\in\Omega\,|\,n+1$

$$\lambda_{n+1}(Q^*) = \begin{cases} \lambda_n(Q^*) & \text{if } Q^*\in\Omega\,|\,n, \\ \mu_Q\,|\,E(Q^*) & \text{if } Q^*\in Q+1 \text{ for some } Q\in(\Omega\,|\,n)\setminus(\Omega\,|\,n-1). \end{cases}$$

It is now clear that $\lambda_{n+1}: \Omega\,|\,n+1 \to M^+(X)$ is well-defined and has (1∗), (2), (3∗). Thus by induction we have established the assertion.

Let $\mathscr{P}$ be the property of the path bundles $\Omega(\mathscr{D})$. We denote by $\Omega(\mathscr{D})\,|\,\mathscr{P}$ the totality of path bundles with $\mathscr{P}$. That is,

$$\Omega(\mathscr{D})\,|\,\mathscr{P} = \{Q\in\Omega(\mathscr{D}): Q \text{ has the property } \mathscr{P}\}.$$

The typical properties are the following: with $\mathscr{K}$ contained in $\mathscr{D}$.

$$(2.1)\begin{cases} (1)\ \operatorname{card}(Q)\leqq n; \\ (2)\ Q\supset Q^* \text{ (the prolongations of } Q^*); \\ (3)\ [\mathscr{B}(Q)\supset\mathscr{K}]\wedge[(\mathscr{C}(Q)=\varnothing)\vee(\mathscr{C}(Q)\cap\mathscr{K}\neq\varnothing)] \text{ (path bundles} \\ \quad\ \text{checked by the “boundary” of the compact set } \bigcap\{V: V\in\mathscr{K}\}), \end{cases}$$

where $\vee$ (resp. $\wedge$) implies “or” (resp. “and”). These three properties will be denoted by the letters n, Q^* and $\mathscr{K}$ respectively. Let us note the identity $\Omega\,|\,(\mathscr{K}\wedge Q^*) = (\Omega\,|\,\mathscr{K})\cap(\Omega\,|\,Q^*)$, where $\Omega=\Omega(\mathscr{D})$.

LEMMA 2.7. *Let $\mathscr{D}$ be a finite subset of $\mathscr{F}_q$ and let $\Omega=\Omega(\mathscr{D})$ be the path bundles on $\mathscr{D}$. For any $Q\in\Omega$, if $\mathscr{K}$ is a subset of $\mathscr{B}(Q)$ satisfying $Q\notin\Omega\,|\,(Q\wedge\mathscr{K})$, then the collection $\{\Omega\,|\,(Q^*\wedge\mathscr{K}): Q^*\in Q+1\}$ given a partition of $\Omega\,|\,(Q\wedge\mathscr{K})$ into pairwise disjoint subsets.*

If $\mathscr{K}$, $\mathscr{L}$ are subsets of $\mathscr{D}$ such that $\mathscr{K}\supset\mathscr{L}$, then the collection $\{\Omega\,|\,(\mathscr{L}\wedge Q): Q\in\Omega\,|\,\mathscr{K}\}$ gives a decomposition of $\Omega\,|\,\mathscr{L}$ into pairwise disjoint subsets.

PROOF. For the first assertion we note that every Q^* of $Q+1$ admits

$$\Omega \mid (\mathcal{K} \wedge Q^*) = \Omega \mid Q^* \cap \Omega \mid \mathcal{K} \subset \Omega \mid Q \cap \Omega \mid \mathcal{K} = \Omega \mid (Q \wedge \mathcal{K}),$$

because $\Omega \mid Q^* \subset \Omega \mid Q$. Now, assume that $\Omega \mid (Q_1 \wedge \mathcal{K}) \cap \Omega \mid (Q_2 \wedge \mathcal{K}) \ni Q_0$ for some Q_1, Q_2 of $Q + 1$. Then it follows from the definition of path bundles that $Q_0 \mid \mathrm{card}(Q_j) = Q_j$ $(j = 1, 2)$. So we get $Q_1 = Q_2$, since $\mathrm{card}(Q_j)$ are identical. On the other hand, for any $Q_1 \in \Omega \mid (Q \wedge \mathcal{K})$, the set $Q + 1$ contains $Q^* = Q_1 \mid (\mathrm{card}(Q) + 1)$, because $\mathcal{C}(Q) \neq \varnothing$ and $\mathcal{C}(Q) \cap \mathcal{K} = \varnothing$ in this case. From the fact $Q_1 \in \Omega \mid Q^*$, we have $Q_1 \in \Omega \mid (Q^* \wedge \mathcal{K})$, $Q^* \in Q + 1$.

For the second assertion assume that $\Omega \mid (Q_1 \wedge \mathcal{L}) \cap \Omega \mid (Q_2 \wedge \mathcal{L}) \ni Q$ for some Q_1, Q_2 of $\Omega \mid \mathcal{K}$. Since $Q_j = Q \mid \mathrm{card}(Q_j)$ $(j = 1, 2)$, either of the inclusions $Q_1 \subset Q_2$ or $Q_2 \subset Q_1$ must hold. We may assume that $Q_1 \subset Q_2$. If $Q_1 \neq Q_2$, then between final terms $(\mathcal{B}(Q_j), \mathcal{C}(Q_j))$ of Q_j $(j = 1, 2)$, the relations $\mathcal{B}(Q_2) \subset \mathcal{B}(Q_1) \setminus \mathcal{C}(Q_1)$, $\mathcal{C}(Q_1) \neq \varnothing$ hold by (3) in Definition 2.3. On the other hand, $\mathcal{K}$ is contained in $\mathcal{B}(Q_j)$ $(j = 1, 2)$ by the definition of the property $\mathcal{K}$. But this is impossible, because $\mathcal{C}(Q_1) \cap \mathcal{K} \neq \varnothing$ in the present situation. Hence we get $Q_1 = Q_2$.

Conversely if Q^* belongs to $\Omega \mid \mathcal{L}$, $\mathcal{C}(Q^*)$ satisfies either of the relations $\mathcal{C}(Q^*) = \varnothing$ or $\mathcal{C}(Q^*) \cap \mathcal{L} \neq \varnothing$ by (3) in (2.1). Call $Q^* = \{(\mathcal{B}_j, \mathcal{C}_j)\}_{j=1}^{n}$ and let k be the greatest number among j that satisfy $\mathcal{K} \subset \mathcal{B}_j$. Since $\mathcal{B}_1 = \mathcal{D}(\supset \mathcal{K})$ by (1) in Definition 2.3, such k exists. In the case $k < n$, we have $C_k \cap \mathcal{K} \neq \varnothing$, because $\mathcal{K} \not\subset \mathcal{B}_{k+1} = \mathcal{B}_k \setminus \mathcal{C}_k$ by (3) in Definition 2.3. Moreover if $k = n$, we get $\mathcal{C}_k = \mathcal{C}(Q^*) = \varnothing$ or $(\mathcal{C}_k \cap \mathcal{K}) \supset (\mathcal{C}_k \cap \mathcal{L}) \neq \varnothing$. Accordingly it turns out that the path bundle $Q = Q^* \mid k$ belongs to $\Omega \mid \mathcal{K}$, and so we find that $Q^* \in \Omega \mid (Q \wedge \mathcal{L})$, $Q \in \Omega \mid \mathcal{K}$.

LEMMA 2.8. *Let $\mathcal{D}$ be a finite subset of $\mathcal{F}_q$. Every tree λ along $\Omega = \Omega(\mathcal{D})$ with base point q has the following properties*:

(1) *for each subset $\mathcal{K}$ of $\mathcal{D}$, if Q of Ω satisfies $\mathcal{K} \subset \mathcal{B}(Q)$, then*

$$\lambda(Q) \prec \sum_{Q^* \in \Omega \mid (\mathcal{K} \wedge Q)} \lambda(Q^*);$$

(2) *for any two subsets $\mathcal{K}$, $\mathcal{L}$ of $\mathcal{D}$ satisfying $\mathcal{L} \subset \mathcal{K}$, we have*

$$\delta_q \prec \sum_{Q \in \Omega \mid \mathcal{K}} \lambda(Q) \prec \sum_{Q \in \Omega \mid \mathcal{L}} \lambda(Q);$$

(3) *for any $\mathcal{K}$, $\mathcal{L} \subset \mathcal{D}$, if $\bigcap\{V : V \in \mathcal{K}\} = \bigcap\{V : V \in \mathcal{L}\}$, then we get*

$$\sum_{Q \in \Omega \mid \mathcal{K}} \lambda(Q) = \sum_{Q \in \Omega \mid \mathcal{L}} \lambda(Q) = \sum_{Q \in (\Omega \mid \mathcal{K}) \cap (\Omega \mid \mathcal{L})} \lambda(Q).$$

PROOF. The proof of (1) is by induction on $\mathrm{card}(\mathcal{B}(Q))$. We first consider the case $\mathrm{card}(\mathscr{B}(Q)) = \mathrm{card}(\mathscr{K})$, i.e. $\mathscr{B}(Q) = \mathscr{K}$. Since $\mathscr{B}(Q)$ contains $\mathscr{C}(Q)$, Q must belong to $\Omega \,|\, \mathscr{K}$ by (3) in (2.1). We require that $\Omega \,|\, (\mathscr{K} \wedge Q) = \{Q\}$. Indeed, if there exists a path bundle Q^* such that $Q \subsetneqq Q^*$ (or equivalently if $\mathscr{C}(Q) \neq \varnothing$), the subset $\mathscr{B}(Q^*)$ of $\mathscr{B}(Q) \setminus \mathscr{C}(Q)$ does not contain $\mathscr{K}$, because $\mathscr{K} \cap \mathscr{C}(Q) \neq \varnothing$ in this case. Hence $\Omega \,|\, (Q \cap \mathscr{K}) \not\ni Q^*$, i.e. $\Omega \,|\, (Q \wedge \mathscr{K}) = \{Q\}$. In the case $\mathscr{C}(Q) = \varnothing$, it is easy to check that $Q \in \Omega \,|\, (Q \wedge \mathscr{K}) \subset \Omega \,|\, Q = \{Q\}$. So in both cases, we have (1).

Next, assume that (1) is true for all Q with properties $\mathscr{K} \subset \mathscr{B}(Q)$ and $\mathrm{card}(\mathscr{B}(Q)) < k + \mathrm{card}(\mathscr{K})$ $(k \geqq 1)$. Take any Q that satisfies $\mathscr{K} \subset \mathscr{B}(Q)$ and $\mathrm{card}(\mathscr{B}(Q)) = k + \mathrm{card}(\mathscr{K})$. In the case $Q \subset \Omega \,|\, \mathscr{K}$, i.e. $\mathscr{C}(Q) = \varnothing$ or $\mathscr{C}(Q) \cap \mathscr{K} \neq \varnothing$, we now obtain easily that $\Omega \,|\, (Q \wedge \mathscr{K}) = \{Q\}$. Hence (1) is valid for such Q. Suppose Q is not in $\Omega \,|\, \mathscr{K}$. From (8) in Definition 2.3 and by the fact $\mathscr{C}(Q) \neq \varnothing$, $\mathscr{C}(Q) \cap \mathscr{K} = \varnothing$, any $Q^* \in Q + 1$ admits the relation

$$\mathscr{K} \subset \mathscr{B}(Q^*) = \mathscr{B}(Q) \setminus \mathscr{C}(Q), \quad \mathrm{card}(\mathscr{B}(Q^*)) < k + \mathrm{card}(\mathscr{K}).$$

Therefore we get by induction hypothesis and from Lemma 2.7 that

$$\begin{aligned} \lambda(Q) &\prec \sum_{Q^* \in Q+1} \lambda(Q^*) \\ &\prec \sum_{Q^* \in Q+1} \left\{ \sum_{Q^0 \in \Omega | (Q^* \wedge \mathscr{K})} \lambda(Q^0) \right\} \\ &= \sum_{Q^* \in \Omega | (Q \wedge \mathscr{K})} \lambda(Q^*). \end{aligned}$$

Accordingly we conclude (1) by induction.

Property (2) is immediate from (1) and Lemma 2.7. Indeed, for two subsets $\mathscr{K}$, $\mathscr{L}$ of $\mathscr{D}$ such that $\mathscr{L} \subset \mathscr{K}$, we have

$$\begin{aligned} \sum_{Q \in \Omega | \mathscr{K}} \lambda(Q) &\prec \sum_{Q \in \Omega | \mathscr{K}} \left\{ \sum_{Q^* \in \Omega | (\mathscr{L} \wedge Q)} \lambda(Q^*) \right\} \\ &= \sum_{Q \in \Omega | \mathscr{L}} \lambda(Q). \end{aligned}$$

In particular, by (2) in Definition 2.5,

$$\delta_q \prec \sum_{\mathrm{card}(Q) = 1} \lambda(Q) = \sum_{Q \in \Omega | \mathscr{D}} \lambda(Q) \prec \sum_{Q \in \Omega | \mathscr{K}} \lambda(Q).$$

For (3) we first treat the case $\mathscr{L} \subset \mathscr{K}$. Let Q be any path bundle of $\Omega \,|\, \mathscr{K}$ that satisfies $\Omega \,|\, (Q \wedge \mathscr{L}) \neq \{Q\}$. We show that $\lambda(Q) = 0$ for such Q. In other words, we require that $\lambda(Q)$ vanishes at $Q \in \Omega \,|\, \mathscr{K}$ whenever $\mathscr{C}(Q) \neq \varnothing$ and

$\mathscr{C}(Q) \cap \mathscr{L} = \varnothing$. This will be done if we can show that $E(Q)$ is empty for this Q. Recall that $\mathscr{B}(Q)$ contains $\mathscr{K}$. In particular, $\mathscr{L}$ is included in $\mathscr{B}(Q) \setminus \mathscr{C}(Q)$. This in turn yields

$$E(Q) \subset [\bigcap\{\text{Int } V : V \in \mathscr{B}(Q) \setminus \mathscr{C}(Q)\}] \subset \text{Int}[\bigcap\{V : V \in \mathscr{L}\}]$$

and

$$E(Q) = [\bigcap\{\text{Int } V : V \in \mathscr{B}(Q) \setminus \mathscr{C}(Q)\}] \cap [\bigcap\{\partial V : V \in \mathscr{C}(Q)\}]$$
$$\subset \partial[\bigcap\{V : V \in \mathscr{K}\}].$$

The last inclusion is due to the fact $\mathscr{K} \cap \mathscr{C}(Q) \neq \varnothing$. These imply that $E(Q) = \varnothing$, because $\bigcap\{V : V \in \mathscr{L}\} = \bigcap\{V : V \in \mathscr{K}\}$. Hence we obtain $\lambda(Q) = 0$ whenever $\Omega \mid (\mathscr{L} \wedge Q) \neq \{Q\}$ and $Q \in \Omega \mid \mathscr{K}$. This is nothing but (3) in the present situation. For general $\mathscr{K}, \mathscr{L} \subset \mathscr{D}$, put $\mathscr{M} = \mathscr{K} \cup \mathscr{L}$. Then from the result just proven, it follows that $\lambda(Q) = 0$ if $Q \in \Omega \mid \mathscr{M} \setminus \Omega \mid \mathscr{K}$ or $Q \in \Omega \mid \mathscr{K} \setminus \Omega \mid \mathscr{M}$, etc. Accordingly we conclude that

$$\sum_{Q \in \Omega \mid \mathscr{M}} \lambda(Q) = \sum_{Q \in (\Omega \mid \mathscr{M}) \cap (\Omega \mid \mathscr{K}) \cap (\Omega \mid \mathscr{L})} \lambda(Q) = \sum_{Q \in \Omega \mid \mathscr{L}} \lambda(Q) = \sum_{Q \in \Omega \mid \mathscr{K}} \lambda(Q).$$

Proposition 2.9. *Let $\mathscr{D}$ be any finite subset of $\mathscr{F}_q$ and let λ be any tree along $\Omega(\mathscr{D})$ with base point q. For any subset $\mathscr{K}$ of $\mathscr{D}$ and the compact set $G \subset X$ satisfying $G = \bigcap\{V : V \in \mathscr{K}\}$, if we call $\lambda_G = \Sigma_{Q \in \Omega(\mathscr{D}) \mid \mathscr{K}} \lambda(Q)$, then λ_G is well-defined. The totality of such λ_G forms a consistent family with base point q, which is indexed by the finite set $\{\bigcap\{V : V \in \mathscr{K}\} : \mathscr{K} \subset \mathscr{D}\}$. Furthermore, if the tree λ satisfies* (1$*$) *in Proposition* 2.6, *the terminal measure of the above consistent family is maximal on X in the order $\prec$.*

Proof. With the aid of (3) in Lemma 2.8, the measure λ_G in question is determined only by G: it is independent of the choice of $\mathscr{K} \subset \mathscr{D}$ Hence λ_G is well-defined. Moreover, for each Q of $\Omega \mid \mathscr{K}$, $E(Q)$ is contained in the closed set $\bigcap\{V : V \in \mathscr{B}(Q)\}$, hence in G, because $\mathscr{K} \subset \mathscr{B}(Q)$ by (3) in (2.1). Since $\lambda(Q)$ is carried on $\overline{E(Q)}$, we find that λ_G is supported on G. Call $\mathscr{D}^0 = \{\bigcap\{V : V \in \mathscr{K}\} : \mathscr{K} \subset \mathscr{D}\}$ and we prove that the subset $\{\lambda_G : G \in \mathscr{D}^0\}$ of $M^+(X)$ is a consistent family with base point q. As shown above, each λ_G is carried on G. Next let F, G be any two elements of $\mathscr{D}^0$ satisfying $F \subset G$. Then there exist two subsets $\mathscr{K}, \mathscr{L}$ of $\mathscr{D}$ such that $\mathscr{L} \subset \mathscr{K}$, $F = \bigcap\{V : V \in \mathscr{K}\}$ and $G = \bigcap\{V : V \in \mathscr{L}\}$. By (2) in Lemma 2.8 it turns out that $\delta_q \prec \lambda_F \prec \lambda_G$. Let us verify the inequality $\lambda_F \mid U \leqq \lambda_G$ for the relative interior U of F with respect to G. Suppose a path bundle $Q \in \Omega \mid \mathscr{K}$ satisfies $\mathscr{C}(Q) \neq \varnothing$ and $\mathscr{C}(Q) \cap \mathscr{L} = \varnothing$.

We require that $\overline{E(Q)}$ is included in $F \setminus U$. This is immediate from the relation $\mathscr{L} \subset \mathscr{B}(Q) \setminus \mathscr{C}(Q)$ and $\mathscr{C}(Q) \cap \mathscr{K} \neq \varnothing$, because they imply the following inclusion:

$$E(Q) \subset [\bigcap\{\mathrm{Int}\, V : V \in \mathscr{B}(Q) \setminus \mathscr{C}(Q)\}] \cap [\bigcap\{\partial V : V \in \mathscr{C}(Q)\}]$$
$$\subset (\mathrm{Int}\, G) \cap \partial F.$$

In particular $E(Q) \subset F \setminus U$, and so $\overline{E(Q)} \subset F \setminus U$. Accordingly it turns out that

$$\lambda_F \mid U \leqq \sum_{\substack{Q \in \Omega \mid \mathscr{K} \\ \mathscr{C}(Q) = \varnothing \text{ or } \mathscr{C}(Q) \cap \mathscr{L} \neq \varnothing}} \lambda(Q) = \sum_{Q \in (\Omega \mid \mathscr{K}) \cap (\Omega \mid \mathscr{L})} \lambda(Q)$$
$$\leqq \sum_{Q \in \Omega \mid \mathscr{L}} \lambda(Q) = \lambda_G.$$

Therefore we conclude that $\{\lambda_G : G \in \mathscr{D}^0\}$ is a consistent family with base point q.

Finally suppose λ has property (1$*$) in Proposition 2.6. Then for each Q of $\Omega \mid \varnothing$ the set $E(Q)$ is identical with the open set $\bigcap\{\mathrm{Int}\, V : V \in \mathscr{B}(Q)\}$, because $\mathscr{C}(Q)$ is empty in this case. On the other hand, $\lambda(Q)$ is maximal on the compact set $\bigcap\{V : V \in \mathscr{B}(Q)\}$ and carried on its interior. Hence we get, by Theorem 2.1, that $\lambda(Q)$ is maximal on X. In particular, the measure $\Sigma_{Q \in \Omega \mid \varnothing} \lambda(Q)$ is maximal on X in the order $\prec$, which is identical with the terminal measure λ_X of the consistent family. Indeed, $X = \bigcap\{V : V \in \varnothing\}$.

THEOREM 2.10. *Let X be a compact Hasudorff space and let J be a convex subcone of $C_R(X)$ with the three properties* (a), (b), (c) *mentioned earlier. Then for any point q of X there exists a consistent family, indexed by $\mathscr{F}_q$, whose terminal measure is carried on the J-Shilov boundary. In the case that the measures maximal on X and dominating δ_q in the order $\prec$ form a compact convex set in the dual of $C_R(X)$, we can take a consistent family, indexed by $\mathscr{F}_q$, whose terminal measure is maximal on X.*

PROOF. Let $\mathscr{D}$ be any finite subset of $\mathscr{F}_q$ satisfying $X \in \mathscr{D}$. By the preceding proposition, we find that there exists a consistent family $\{\lambda_G^{\mathscr{D}} : G \in \mathscr{D}\}$ with base point q whose terminal measure $\lambda_X^{\mathscr{D}}$ is maximal on X. Since the totality of $\mathscr{D}$ has the natural order of set theoretic inclusion, there exists an ultra filter I, defined on the set of all $\mathscr{D}$, which is finer than the natural order. For each $G \in \mathscr{F}_q$ we denote by λ_G the vague limit of $\lambda_G^{\mathscr{D}}$ along I. Note that λ_G is well-defined, because G is contained in any $\mathscr{D}$ large enough. Here we require that $\{\lambda_G : G \in \mathscr{F}_q\}$ is the desired consistent family. Indeed, for any nonempty finite set $\mathscr{K} \subset \mathscr{F}_q$, $\{\lambda_G^{\mathscr{D}} : G \in \mathscr{K}\}$ is consistent, whenever $\mathscr{D} \supset \mathscr{K}$. Applying Theorem 2.2, we see that $\{\lambda_G : G \in \mathscr{K}\}$ is a consistent family with base point q. Since $\mathscr{K}$ is arbitrary, we conclude that $\{\lambda_G : G \in \mathscr{F}_q\}$ is consistent. Moreover any $\lambda_X^{\mathscr{D}}$ is maximal on X, and so supported on the J-Shilov boundary. Hence λ_X is carried on the J-Shilov boundary. The final assertion is now obvious.

3. Uniform algebras

Let X be the maximal ideal space of an arbitrary uniform algebra A, and let J_A denote the totality of continuous A-subharmonic functions on X. J_A consists of functions in $C_R(X)$ which are uniformly approximated by functions;

$$\bigvee_{j=1}^{n} c_j \log|f_j|, \quad R \ni c_j \geqq 0, \quad A \ni f_j \quad (1 \leqq j \leqq n < \infty).$$

The convex cone J_A satisfies three conditions (a), (b), (c) required of J in the preceding section: about (a), (b), everything is trivial. The sheaf structure of J_A is due to the localization principle established by T. W. Gamelin and N. Sibony [7] (cf. [6]). Concerning the J_A-Shilov boundary, it is known, and easy to check, that the Shilov boundary for A coincides with the J_A-Shilov boundary. Also we note that any $\mu \in M^+(X)$ is a Jensen measure for $q \in X$ if and only if it satisfies $\delta_q \prec \mu$. Hence we get the following.

THEOREM 3.1. *Let X be the maximal ideal space of any uniform algebra A. Then for any point q of X there exists a consistent family $\{\lambda_G : G \in \mathscr{F}_q\}$ of Jensen measures for q whose terminal measure λ_X is carried on the Shilov boundary for A. In this case, we use the cone J_A to define the partial order $\prec$ over $M^+(X)$.*

Let D^n be the unit polydisc in n-dimensional complex Euclidean space and let J_n be the set of functions f of $C_R(\overline{D^n})$, each of which is n-subharmonic in D^n, that is, the restriction $f \mid D^n$ of f onto D^n is subharmonic in each variable separately (cf. W. Rudin [9]). It is easy to verify that J_n has (a), (b), (c) required of J as a convex cone on the compact Hausdorff space $\overline{D^n}$. The J_n-Shilov boundary is identical with the torus T^n, or in other words the distinguished boundary of D^n, which in turn coincides with the Shilov boundary for the polydisc algebra $A(D^n)$. Also, for a real-valued continuous funtion g on T^n there exists a unique function in $J_n \cap (-J_n)$ which is equal to g on T^n. In other words, the Banach space $J_n \cap (-J_n)$ with sup norm is a Banach lattice. Hence for each point q of $\overline{D^n}$ the Dirac measure δ_q is dominated by a unique positive measure on the torus T^n with respect to the order $\prec$ induced by J_n. In particular δ_0, $0 \in D^n$, is dominated by the Haar measure $(d\theta/2\pi)^n$ on T^n in the order $\prec$. Moreover, any consistent family with respect to J_n is also a consistent family of Jensen measures associated with $A(D^n)$, since $J_{A(D_n)} \subset J_n$.

THEOREM 3.2. *The Haar measure $(d\theta/2\pi)^n$ on the torus T^n is the terminal measure of some consistent family $\{\lambda_G : G \in \mathscr{F}_0\}$, $0 \in D^n$, which is associated with the cone J_n. Furthermore this family is also a consistent family of Jensen measures for 0 with respect to the cone of $A(D^n)$-subharmonic functions.*

4. Compact convex sets

Let X be any compact convex set in some locally convex topological vector space, and let the letter J denote the totality of continuous convex functions on X. It is known that the convex cone J has (a), (b), (c) mentioned in Section 2 (cf. A. Zygmund [10]). The partial order on $M^+(X)$ induced by J is denoted by the symbol $\prec$. If probability measures μ, ν in $M^+(X)$ satisfy $\mu \prec \nu$, then μ, ν have a common barycenter in X. Also, a point q of X is the barycenter of μ if and only if it satisfies $\delta_q \prec \mu$ (cf. [1]).

The purpose in this section is to prove that every probability measure μ in $M^+(X)$ is the terminal measure of some consistent family $\{\lambda_G : G \in \mathscr{F}_q\}$, associated with the order $\prec$, whose base point q is the barycenter of μ. For a nonempty finite set D of points on X, the following measures:

$$\sum_{y \in D} c_y \delta_y, \quad R \ni c_y \geqq 0, \quad \sum_{y \in D} c_y = 1,$$

will be called probability simple measures. It is known that any probability measure $\mu \in M^+(X)$ with barycenter q is included in the vague closure in $M^+(X)$ of probability simple meaures with common barycenter q. Therefore with the aid of theorem 2.2, we have only to treat probability simple measures. Here let us agree to use the symbol $[w, z]$ (resp. (w, z)) for the segment (resp. free segment) on X connecting w, z in X.

LEMMA 4.1. *Every probability simple measure $\mu = \Sigma_{y \in D} c_y \delta_y$, on X is the terminal measure of some consistent family $\{\lambda_G : G \in \mathscr{F}_q\}$, associated with the order $\prec$, whose base point q is equal to the barycenter of μ.*

PROOF. The proof is by induction on card(D). If D is a singleton $\{q\}$, then the set of measures $\lambda_G = \delta_q$, $G \in \mathscr{F}_q$, is clearly the desired family.

Next, assume that the assertion is true for any probability simple measures μ as above with card(D) less than n $(1 < n < \infty)$. Let D be a subset of X which consists of n distinct points, and let $\mu = \Sigma_{y \in D} c_y \delta_y$ be a probability simple measure such that $0 < c_y < 1$ for each $y \in D$. We first take a point p from D and call $\nu = \Sigma_{y \in D \setminus \{p\}} c_y \delta_y / (1 - c_p)$. By the induction hypothesis, ν is the terminal measure of a consistent family $\{\lambda_G^0 : G \in \mathscr{F}_r\}$, where r denotes the barycenter of ν. Applying this family, we manufacture the consistent family $\{\lambda_G : G \in \mathscr{F}_q\}$ such that $\lambda_X = \mu$. In the case $p = q$, or equivalently if $p = r$, the set of measures $\lambda_G = c_p \delta_p + (1 - c_p)\lambda_G^0$ gives the desired family by Theorem 2.2. So, in the sequel, we suppose $p \neq q$ and $r \neq q$.

When $G \in \mathscr{F}_q$ does not contain q in its interior, we put $\lambda_G = \delta_q$. If G satisfies $\operatorname{Int} G \ni q$, we first take the connected component (w, z) of $(\operatorname{Int} G) \cap (p, r)$ which contains q. Note that this is possible, since q belongs to (p, r) by the fact $\delta_q \prec c_p \delta_p + (1 - c_p)\delta_r$. We may suppose that p, w, z, r are placed in this order. Call $I_G = [w, z]$. Since I_G contains q, there exists a unique convex combination $m\delta_w + (1 - m)\ \delta_z$ whose barycenter is equal to q. We put $\lambda_G = m\delta_w + (1 - m)\delta_z$ if $z \neq r$ and $\lambda_G = m\delta_w + (1 - m)\lambda_G^0$ otherwise. Here let us note that if $z = r$, the set G belongs to $\mathscr{F}_r$, and also, that $\lambda_X = c_p \delta_p + (1 - c_p)\lambda_X^0 = \mu$.

The family $\{\lambda_G : G \in \mathscr{F}_q\}$ thus defined gives the desired consistent family. Clearly each λ_G is carried on G. Also, by the very definition of λ_G it follows easily that $\delta_q \prec \lambda_G$ and $\lambda_X = \mu$. Therefore we have only to prove that $\lambda_F \prec \lambda_G$ and $\lambda_F \mid U \leqq \lambda_G$ for $F, G \in \mathscr{F}_q$ such that $F \subset G$, where U is the relative interior of F with respect to G. If q is not in $\operatorname{Int} F$ then $\lambda_F = \delta_q$, and either of the following relations must be valid: $\lambda_G = \delta_q$ or $U \not\ni q$. Hence we get that $\lambda_F \prec \lambda_q$ and $\lambda_F \mid U \leqq \lambda_G$ in this case.

Assume that $q \in \operatorname{Int} F$. Furthermore we call $I_F = [b, e]$ and $I_G = [w, z]$. If $z \neq r$, then λ_F and λ_G can be written as $\lambda_F = k\delta_b + (1 - k)\delta_e$ and $\lambda_G = m\delta_w + (1 - m)\delta_z$ with k, m in $(0, 1)$. Since each $f \in J$ is convex on $[w, z]$, we have $kf(b) + (1 - k)f(e) \leqq mf(w) + (1 - m)f(z)$, i.e. $\lambda_F \prec \lambda_G$. This is due to the fact that two measures have the same barycenter and $I_F \subset I_G$. In the case $e \neq z$, U cannot include e. The same is true for b and w. On the other hand, if $e \neq z$ and $b = w$, then we have $k \leqq m$, and so $\lambda_F \mid U \leqq \lambda_G$. In the similar way, we can check the inequality $\lambda_F \mid U \leqq \lambda_G$ in each case.

When z is equal to r, it is easy to verify that $m\delta_w + (1 - m)\delta_r \prec \lambda_G$. If further e is different from r, we get that $\lambda_F \prec m\delta_w + (1 - m)\delta_r \prec \lambda_G$ and $\lambda_F \mid U \leqq m\delta_w + (1 - m)\delta_r$ by the same argument as above. In particular, we have $\lambda_F \mid U \leqq m\delta_w \leqq \lambda_G$. In the case $e = r = z$, λ_F and λ_G are expressed as $\lambda_F = k\delta_b + (1 - k)\lambda_F^0$, $\lambda_G = m\delta_w + (1 - m)\lambda_G^0$. So we get $\lambda_F \mid U \leqq \lambda_G$, because $\lambda_F^0 \mid U \leqq \lambda_G^0$ and $m \leqq k$ (the equality holds if and only if $b = w$}. It is easy to check that $\lambda_F \prec \lambda_G$ when $b = w$. If, on the other hand, $b \neq w$, the barycenter of

$(m\delta_w + (k-m)\lambda_G^0)$ is equal to b. Hence we have $k\delta_b \prec (m\delta_w + (k-m)\lambda_G^0)$, $(1-k)\lambda_F^0 \prec (1-k)\lambda_G^0$, and so $\lambda_F \prec \lambda_G$. These yield that $\{\lambda_G : G \in \mathscr{F}_q\}$ is the desired consistent family. Thus by induction we conclude the assertion.

THEOREM 4.2. *Every probability measure $\mu \in M^+(X)$ is the terminal measure of some consistent family $\{\lambda_G : G \in \mathscr{F}_q\}$, associated with the order $\prec$, whose base point q is equal to the barycenter of μ.*

In such a situation, all the results in [8] are still valid except for the Riesz type estimate about conjugate functions: let $\mu \in M^+(X)$ be a probability measure with barycenter q and let $\{\mu_G : G \in \mathscr{F}_q\}$ be a consistent family such that $\mu_X = \mu$. For notational convenience, we put $\mu[f \leqq t] = \mu_G$ if $t > f(q)$ and $\mu\{f \leqq t] = \delta_q$ otherwise, where $f \in J$, $G = \{y \in X : f(y) \leqq t\}$ and $t \in R$. Also we call $\mu[f = t] = \mu[f \leqq t] \,|\, \{y \in X : f(y) = t\}$ for $t > f(q)$, $\mu[f = t] = \delta_q$ if $t \leqq f(q)$, and $\mu[f < t] = \mu[f \leqq t] - \mu[f = t]$. By Definition 1.1, $\mu[f < t]$ is absolutely continuous with respect to $\mu = \mu_X$ and its density satisfies $0 \leqq d\mu[f < t]/d\mu \leqq d\mu[f < s]/d\mu \leqq 1$, $t \leqq s$, a.e. μ. Note that this density can be regarded as a measurable function on the product measure $d\mu \cdot dt$ via the trivial modification, where dt is the Lebesgue measure on R. The maximal function $M_B(f^p)$ of Brownian motion type is defined for each nonnegative function $f \in J$: with p in the interval $(0, \infty)$

$$M_B(f^p)(y) = \int_0^\infty pt^{p-1}\{1 - d\mu[f < t]/d\mu(y)\}dt.$$

If f^r belongs to J for some $r > 0$ and if $p > r$, we have $\int M_B(f^p)d\mu \leqq (p/p-r)^{p/r} \int f^p d\mu$. This inequality was established by D. L. Burkholder, R. F. Gundy and M. L. Silverstein [2] in the case of Brownian maximal functions associated with classical Hardy spaces. The proof is immediate from a version of Doob's maximal inequality (cf. [4]). In the present situation Doob's inequality takes the following form (cf. [8]): since $f^r \in J$ and $\mu[f = t^c] \prec \mu_X - \mu[f < t^c]$, $c = r/p$, we have

$$t^{cr}\mu[f = t^c](X) \leqq \int f^r d(\mu[f = t^c]) \leqq \int f^r d(\mu_X - \mu[f < t^c]).$$

Furthermore, the Fefferman–Stein type estimate [5] holds: if $f, h \in J \cap (-J)$ and $(hf)(q) = 0$, we get from the similar argument as in Theorem 5.7 [8] that

$$\left|\int fh d\mu\right| \leqq \sqrt{2}\,\eta(h) \int M_B(|f|)d\mu \leqq 2\sqrt{2}\,\eta(h)\left\{1 + \int |f| \log^+ |f| \,d\mu\right\},$$

where $\eta(h)$ denotes Garsia "norm":

$$\eta(h) = [sup\{-h^2(y) - (\widecheck{-h^2})(y) : y \in X\}]^{1/2}$$

and the last inequality is due to Zygmund's $L \log L$ estimate (cf. Appendix below).

5. Appendix

Let X be any compact Hausdorff space and let J be a convex cone in $C_R(X)$ with (a), (b), (c) mentioned in Section 2. We use the symbol $\prec$ to denote the partial order over $M^+(X)$ induced by J. For any q of X take an arbitrary consistent family $\{\mu_G : G \in \mathscr{F}_q\}$ associated with $\prec$, and we prove Zygmund's $L \log L$ estimate for maximal functions (cf. [10]).

THEOREM 5.1. *Let $\{\mu_G : G \in \mathscr{F}_q\}$ be an arbitrary consistent family, and let k denote any nonnegative function in J. Then we have*

$$\int M_B(k) d\mu_X \leqq 2\left\{1 + \int k \log^+ k \, d\mu_X\right\},$$

where $\log^+ t = \log(1 \vee t)$.

PROOF. Let ν be the distribution of $k : \nu(t) = \mu_X(\{y \in X : k(y) \leqq t\})$. Applying the well-known formula, we have $\int_1^\infty t \log^+ t \, d\nu(t) = \int k \log^+ k \, d\mu_X$. On the other hand, it follows from Fubini's theorem that

$$\int M_B(k) d\mu_X$$

$$= \int_0^\infty \{1 - \mu[k < t](X)\} dt = \int_0^\infty \mu[k = t](X) dt = 2 \int_0^\infty \mu[k = 2t](X) dt.$$

Since $s\mu[k = s](X) \leqq \int kd(\mu[k = s])$, $s > 0$, we get for any $t > 0$

$$\begin{aligned} \mu[k = 2t](X) &\leqq (1/2t)\left(\int kd(\mu[k \leqq 2t]) - \int kd(\mu[k < 2t])\right) \\ &\leqq (1/2t)\left(\int kd\mu_X - \int kd(\mu[k < 2t])\right) \\ &\leqq (1/2t) \int kd(\mu_X - \mu[k < 2t]). \end{aligned}$$

Note that $\mu[k \leqq 2t] \prec \mu_X$ and $\mu_X - \mu[k < 2t] \geqq 0$. Putting $E = \{y \in X : k(y) \geqq t\}$ we are led to the estimate:

$$\begin{aligned}\mu[k=2t] &\leqq (1/2t)\int_E kd(\mu_X-\mu[k<2t])+(1/2t)\int_{X\setminus E} kd(\mu_X-\mu[k<2t]) \\ &\leqq (1/2t)\int_E kd\mu_X+(1/2t)\int_{X\setminus E} td(\mu_X-\mu[k<2t]) \\ &\leqq (1/2t)\int_E kd\mu_X+(1/2)(\mu_X-\mu[k<2t])(X) \\ &= (1/2t)\int_E kd\mu_X+(1/2)\mu[k=2t](X).\end{aligned}$$

Therefore, applying the distribution ν of k, we obtain

$$\mu[k=2t](X)\leqq(1/t)\int_E kd\mu_X=(1/t)\int_t^\infty sd\nu(s).$$

Accordingly it turns out that

$$\begin{aligned}\int_0^\infty \mu[k=2t](X)dt &\leqq 1+\int_1^\infty \mu[k=2t](X)dt \\ &\leqq 1+\int_1^\infty sd\nu(s)\int_1^s(1/t)dt \\ &= 1+\int_1^\infty s\log^+ s\,d\,\nu(s) \\ &= 1+\int k\log^+ kd\mu_X.\end{aligned}$$

Hence we conclude that

$$\int M_B(k)d\mu_X\leqq 2\left\{1+\int k\log^+ kd\mu_X\right\}.$$

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